Flexible Bayesian Analysis of First Price Auctions Using Simulated Likelihood (Job Market Paper)*

Dong-Hyuk Kim†

October 26, 2009

Abstract

Current nonparametric methods often conduct inference based on estimated bid distributions that cannot be generated by an equilibrium. Then, the analysis results, e.g., policy recommendations, may be suspicious. This actually happens for the sample from the OCS auctions that has been widely used. To provide more reliable inference, we develop a Bayesian method satisfying the equilibrium implications: (i) bidding monotonicity and (ii) density affiliation. We obtain (i) by directly parametrizing the valuation density and (ii) by putting zero prior on every non-affiliated density. Our method allows for a very flexible specification. To handle this highly parametrized model, we use a simulated likelihood. We re-analyze the sample from OCS auctions. Our approach gives significantly different implications for auction design than preivous methods. Since the sample is small, the additional equilibrium information would have a large contribution.

*I am greatly indebted to Keisuke Hirano, my advisor, for his guidance, patience, and encouragement. I am grateful to my professors, Jonah Gelbach, Gautam Gowrisankaran, John Wooders, and Mo Xiao for their invaluable advices and suggestions. I would also like to thank Tong Li for providing me with the data.

†Economics Department at the University of Arizona, Tucson, AZ. Email: donghyuk@email.arizona.edu.

1 Introduction

Auctions are a frequently used market institution for allocating a variety of economic resources, accounting for a significant portion of the U.S. economy. For example, since 1954 the U.S. government has leased approximately thirty thousand tracts for offshore oil and gas development in areas of the Gulf of Mexico. Each of these *Outer Continental Shelf* (OCS) auctions has a trade value which ranges from hundreds of thousands to several million dollars.[1] Reflecting the economic significance of auctions, many economists have extensively investigated this mechanism. In particular, auction thoery has formalized bidding behavior and optimal auction design for certain valuation distribution, while empirical research has focused on estimating the valuation distribution and using these estimates for auction design, e.g., the estimation of optimal reserve prices.

For first price sealed bid auctions with independent private values (IPV), the pioneers in the empirical auction literature specify the valuation distribution using strong parametric assumptions.[2] However, a flexible specification is often preferred because inference exclusively relies on the shape of valuation distribution. For this reason Guerre, Perrigne, and Vuong (2000) indirectly recover the valuation density by inverting the estimated bid distribution. Li, Perrigne, and Vuong (2002) generalize this method to affiliated private value auctions (APV). These indirect approaches have been widely used because they are fully flexible and computationally simple.

However, the indirect methods do not exploit equilbrium implications, *(i) bidding monotonicity* and *(ii) density affiliation*. Thus, their analysis may lack these equilibrium properties for a finite sample. This problem is empirically important for the datasets of interest to researchers and policymakers. For instance, Li, Perrigne, and Vuong (2003) propose an optimal reserve price for the OCS auctions using a bid sample of 217 observations. We find that their policy recommendations are based on an estimated bid

[1] For more information, visit the website of the Offshore Energy and Minerals Management (OEMM) at `www.mms.gov/offshore/`. As can be found there, similar offshore oil and gas auctions have being held in other areas such as Alaska, Pacific, and Atlantic. The governement also has auctioned off various onshore mineral rights, timber rights, magnetic spectrums, etc. In private sector, countlessly many commodities are auctioned on/off-line.

[2] See Donald and Paarsch (1993, 1996), Laffont, Ossard, and Vuong (1995), and Li and Vuong (1997).

density that cannot be generated by an equilibrium. Thus, policymakers may suspect the validity of such policy implications.

Motivated by this, we develop a Bayesian framework that satisfies (i) and (ii) to provide more reliable infernece. Specifically, we directly parametrize the valuation density so that (i) is satisfied for every parameter value and put zero prior on every density that violates (ii). For a reasonably flexible analysis we employ a method of series representation. To handle such a rich specification, we use a simulated likelihood. In particular, we employ a multinomial likelihood defined on a finely discretized sample space. Then, as Flury and Shephard (2008) discuss, we can obtain an exact posterior even with a finite number of simulation draws.[3] Note that the simulation error would typically inflate the asymptotic variance of other simulation methods.

We revisit the sample from the OCS auctions that Li, Perrigne, and Vuong (2003) have analyzed. We use a prior specification that controls the tail behavior because the optimal reserve price can be unreasonably large for a thick tail. Our methodolgy simulates the posterior distribution of the valuation density for the OCS wildcat auctions. The resulting bid density estimate fits the data very well, but it does not explain closely the outliers because of the prior. We select a reserve price to maximize the seller's future revenue using the Bayesian decision method introduced by Kim (2008). We find that a reserve price of $462 per acre is optimal given our likelihood and prior. According to our counterfactual analysis, this price increases the predictive revenue for each tract by $262,414 relative to the nonbinding reserve for each trade. Note also that our choice is drastically different from the value of $273 obtained by Li, Perrigne, and Vuong (2003), indicating the importance of equilibrium implications.

2 Auction Models and Empirical Method

This section defines the APV/IPV auctions, develops our empirical methodology, and discusses its advantages.

[3]See Andrieu, Doucet, and Holenstein (2007), Andrieu, Doucet, and Roberts (2007). Note that the information loss from the discretization can be minimal, if we use small bins.

2.1 First Price Auctions under the APV/IPV Paradigm

Consider $N \geq 2$ risk neutral bidders in an auction with a reserve price ρ. The bids are collected simultaneously and the bidder with the highest bid obtains the auctioned item at the price equal to his own bid. Let $(v_1, \ldots, v_N) \in \Re_+^N$ be a vector of valuations drawn from a joint distribution F assumed to be absolutely continuous with density f. Each bidder $i = 1, \ldots, N$, after observing his own valuation v_i, bids b_i to maximize his expected utility, $(v_i - b_i)\Pr\left(b_i > \max_{j \neq i} b_j\right)$. The number of bidders N, reserve price ρ, and the valuation distribution F are common knowledge among the bidders.

The auction is said to be under the APV paradigm if the valuations $v_1, \cdots, v_N$ are affiliated. Informally, affiliation implies that if some elements of $(v_1, \cdots, v_N)$ are large, others are more likely to be also large. (See Milgrom and Weber (1982) for a rigorous treatment.) Hence, each bidder gets some information on the distribution of other bidders' valuation from his own valuation and takes it into account to bid optimally. We assume that F is exchangeable so that every bidder is ex-ante identical. This auction mechanism implies a symmetric game of incomplete information among the bidders for which the Bayesian Nash equilibrium with the strictly increasing bidding function $\beta(\cdot|\rho, F)$ is uniquely determined as shown by Milgrom and Weber (1982). Let $f_{y_i|v_i}(\cdot|\cdot)$ be the conditional density of $y_i := \max_{j \neq i} v_j$ given v_i. If $v \geq \rho$, the equilibrium bidding function is given by

$$\beta(v|\rho, F) := v - \int_\rho^v \exp\left\{-\int_\alpha^v \frac{f_{y_1|v_1}(u|u)}{\int_0^u f_{y_1|v_1}(t|u)dt} du\right\} d\alpha \tag{1}$$

Otherwise, any value strictly less than ρ is optimal. Note that the IPV paradigm is a special case of the APV paradigm in which $v_1, \cdots, v_N$ are independent. Under the IPV paradigm, Equation (1) simplifies to

$$\beta(v|\rho, F) := v - \int_\rho^v \left\{\frac{F(\alpha)}{F(v)}\right\}^{N-1} d\alpha \tag{2}$$

where $F(\cdot)$ is the marginal distribution of an individual valuation, with a slight abuse of notation.[4]

[4]Under the IPV paradigm with the exchangeability assumption, the joint valuation distribution is the prod-

Suppose we observe a random sample of T auctions with the common valuation distribution F and a zero reserve price. Let $\beta(\cdot|F) := \beta(\cdot|0, F)$. We assume that for each auction, the N bidders follow the equilibrium bidding function $\beta(\cdot|F)$. Let $z := \left\{(b_{1,t}, \ldots, b_{N,t})\right\}_{t=1}^{T}$ denote the data set. From these data we want to learn F.

2.2 Bayes with Simulated Likelihood

The indirect approaches have been mostly employed for this purpose, because they are flexible and computationally simple. But, they do not use all the available information. For example, their bid density estimates often imply a nonincreasing inverse bidding function. To impose the monotonicity, we directly parametrize the valuation density using a flexible specification $f(\cdot|\theta)$: $\beta(\cdot|\theta)$ is always strictly increasing. (We discuss about other additional information soon.)

Then, a computational difficulty may arise. The bid density is given by

$$g\left(b_1, \ldots, b_N|\theta\right) = \left\{ \frac{f\left(\beta^{-1}(b_1|\theta), \ldots, \beta^{-1}(b_N|\theta)\middle|\theta\right)}{\prod_{i=1}^{N} \beta'\left(\beta^{-1}(b_i|\theta)\middle|\theta\right)} \right\} \cdot 1\left\{\max(b_1, \ldots, b_N) \le \bar{b}(\theta)\right\} \qquad (3)$$

where $\bar{b}(\theta) := \lim_{v\to\infty} \beta(v|\theta)$ and $1\{A\} = 1$ if A is true, otherwise, zero. If we use a flexible specification $f(\cdot|\theta)$ as below, (3) lacks a closed form expression. Hence, we would need to numerically evaluate $\beta^{-1}(\cdot|\theta)$ at each data point for many parameter values. This is impractical, because even for a single evaluation of $\beta^{-1}(\cdot|\theta)$, we need to evaluate the triple integrals in (1) repeatedly. Instead, we simulate the equilibrium bids to construct a likelihood.

A Bayesian approach is strongly motivated in the following reasons. First, the upper bound of bid data $\bar{b}(\theta)$ depends on θ. In this case, the maximum likelihood estimator (MLE) is neither efficient nor asymptotically normal.[5] However, Hirano and Porter (2003) show that the Bayesian estimator is efficient for some specifications. In addition, Andrieu, Doucet, and Holenstein (2007) and Andrieu, Doucet, and Roberts (2007) show

uct of N identical marginals. i.e., $F(v_1, \cdots, v_N) = \prod_{i=1}^{N} F(v_i)$. This abuse of notation should not lead to any confusion.

[5]Though the pseudo MLE of Donald and Paarsch (1993) is asymptotically normal, it is not efficient and hard to apply for a complicated specification.

that an MCMC algorithm converges to the exact posterior distribution if the simulated likelihood is unbiased. This result implies that there may not be a loss of efficiency even with a finite number of simulation draws. Typically, for other simulation methods, simulation not only inflates asymptotic variance but also requires its size to grow.[6] Second, for the APV models we restrict the paprameter space so that every $f(\cdot|\theta)$ is affiliated. The Bayesian methods are very convenient to handle the restricted parameter space problem; it simply puts zero prior over the excluded part of the parameter space. Third, we can control the smoothness and tail behavior by putting a small prior mass, reflecting our beliefs, on the set of too noisy or too thick-tailed densities. As a result, our data analysis, while still flexible, would give a reasonable inference. Note that for a kernel indirect method, there is no formal way to rule out such counterintuitive estimation results. Last, we can use a decision theoretic method for auction desigin as we discuss in section 4.

We briefly review the idea of Andrieu et al., because it would explain how we construct the likelihood and implement an MCMC algorithm. Let $p(\theta)$ and $p(z|\theta)$ denote a prior and a conditional density of data z given θ, and let $p_u(z|\theta)$ be an unbiased estimator for $p(z|\theta)$ constructed by a multidimensional uniform random variable u. If $p(z|\theta)$ were available, we would apply an MCMC algorithm to $p(\theta)p(z|\theta)$, which is proportional to the posterior $p(\theta|z)$. However, suppose that we use $p(\theta)p_u(z|\theta)$, instead. Moreover, $\int p_u(z|\theta)p(u)du = \int p_u(z|\theta)du$, because $p(u) = 1$. Hence, $\int p_u(z|\theta)du = p(z|\theta)$. Thus, $p_u(z|\theta)$ can be viewed as a joint density of (u, z) given θ, say, $p(u, z|\theta)$. Moreover, if we draw a new u at each iteration, this MCMC algorithm generates $(u_1, \theta_1), \ldots, (u_S, \theta_S)$ from

$$p(u, \theta|z) \propto p(\theta)p(u, z|\theta) = p(\theta)p_u(z|\theta)$$

Therefore, $\theta_1, \ldots, \theta_S$ are draws from the exact posterior $p(\theta|z)$. Flury and Shephard (2008) introduce this method to econometrics, which we call in this paper the Bayes with simulated likelihood (BSL). Note that Fernandez-Villaverde, Rudio-Ramirez, and Santos (2006) and Fernandez-Villaverde and Rudio-Ramirez (2007) have employed simulation to form a likelihood.

In order to employ BSL, we first need to construct a likelihood that we can unbiasedly

[6] See Newey and McFadden (1995).

simulate. Then, we must use a new simulation draw at every MCMC step. To obtain an unbiased simulated likelihood, we discretize the empirical sample space into D bins.[7] Then, let y_d denote the number of sample points in the d-th bin and $\pi_d(\theta)$ be the probability of the d-th bin under θ. The true likelihood can be expressed as

$$L(\theta|y) = \prod_{d=1}^{D} \{\pi_d(\theta)\}^{y_d} \tag{4}$$

Let $\widehat{\pi}_d(\theta)$ be the fraction of simulation draws belonging to the d-th bin. Then, we estimate (4) unbiasedly using

$$\widehat{L}(\theta|y) := \prod_{d=1}^{D} \{\widehat{\pi}_d(\theta)\}^{y_d} \tag{5}$$

This approach is very easy to compute and fairly flexible, which is also important, because (5) is to be computed for many parameter candidates and it should not restrict the flexibility of data analysis.[8] Notice that y would have less information than the original bid data z. However, this loss of information can in principle be very small, if we finely discretize the sample space.

2.3 Valuation Density Specification

We discuss our choice for $f(\cdot|\theta)$ for the IPV and APV, separately. The simpler goes first.

IPV case

Since valuations are independent, we model the marginal of the valuation density. We use the specification of Verdinelli and Wasserman (1998). Let $\tilde{F}$ be a simple parametric distribution indexed by μ with density $\tilde{f}$, h be a flexible density on $[0, 1]$ with parameter

[7] Note that, for an APV paradigm, one datum is an observed auction, or its bid profile $(b_{1,t}, \dots, b_{N,t}) \in z$. But, for an IPV model, a single bid, $b_{i,t}$, can be used as a sample point, since all the observed bids are independently and identically distributed.

[8] Note that it might seem to be natural to estimate the theoretical bid density using a kernel method or a method of series over the simulated auctions to construct likelihoods. However, these cannot be our choice, since kernel methods are biased and series methods would not be computationally simple, if flexible.

ψ, and $\theta := (\mu, \psi)$. Now, the specification is given by

$$f(v|\theta) = \tilde{f}(v|\mu)h(\tilde{F}(v|\mu)|\psi) \tag{6}$$

We construct $h(\cdot|\psi)$ as follows. Let $\{\phi_i\}$ denote a sequence of functions on $[0,1]$ such that we can approximate any function on $[0,1]$ using their linear combination $\sum_{i\in I}\psi_i\phi_i$ for some index I and some real numbers $\{\psi_i\}_{i\in I}$. For example, polynomials, splines, or fourier functions can construct such $\{\phi_i\}$. Then,

$$h(x|\psi) \propto \exp\left(\sum_{i\in I}\psi_i\phi_i(x)\right)\cdot 1\{x\in[0,1]\} \tag{7}$$

approximates a density on $[0,1]$. This approach is a method of series. Note that the seminonparametric method of Gallant and Nychka (1987) is an example of series estimation in econometrics.

We employ this particular specification for the following reasons. First, when it extends to the APV, the density affiliation can be simply characterized.[9] Second, using a good $\tilde{f}$ we may reduce the computational cost. To see this, let $x := \tilde{F}(v|\mu)$ and take a logarithm on each side of (6). Then,

$$\log f(v|\theta) = \log\tilde{f}(v|\mu) + \psi_1\phi_1(x) + \psi_2\phi_2(x) + \cdots$$

Thus, $\log\tilde{f}$ approximates first and the additional terms fill out the leftover. Obviously, for a given accuracy, if $\tilde{f}$ is already a good approximation, we would need less components in ψ, implying a lower computational cost. In practice, one may choose $\tilde{f}$ to reflect his beliefs about the form of the true valuation density. For example, if bid data are exponential-like distributed, one could use an exponential with hazard rate μ, believing the valuations may be similarly distributed.

[9] The normal mixture model with Dirichlet process prior of Ferguson (1973), Escobar (1994), and Escobar and West (1995) is widely used for a nonparametric Bayesian analysis. However, we do not employ it for the following reasons. First, there is no easy way to extend it to the APV case, because the density affiliation would be characterized by infinitely many inequality constraints. Second, a normal mixture allows negative values. One might take a log transformation to get around this problem. But, a Bayesian analysis is generally not invariant under a transformation. Third, we use a simulated likelihood. The required modification would not be easy.

To construct (5) we discretize an interval with all the observed bids in z, form the assoicated histogram y, and estimate the bin probabilities using the simulation algorithm given by

1. draw $\tilde{v}_1, \ldots, \tilde{v}_R$ from $f(\cdot|\theta)$.

2. compute $\tilde{b}_1, \ldots, \tilde{b}_R$ by evaluating (2) at each $\tilde{v}_r$, $r = 1, \ldots, R$.

3. Then, $\hat{\pi}_d(\theta) := R^{-1} \sum_{r=1}^{R} 1\left\{\tilde{b}_r \in d\text{-th bin}\right\}$

Appendix A.1 discusses a detailed algorithm for each step.

APV case

We focus on two bidder case ($N = 2$) for clarity. Then, (6) extends to

$$f(v_1, v_2|\theta) = \tilde{f}(v_1|\mu)\tilde{f}(v_2|\mu)h\left(\tilde{F}(v_1|\mu), \tilde{F}(v_2|\mu)\middle|\psi\right) \tag{8}$$

and (7) turns to

$$h(x_1, x_2|\psi) \propto \exp\left(\sum_{i\in I}\sum_{j\in I}\psi_{i,j}\phi_i(x_1)\phi_j(x_2)\right)\cdot 1\left\{(x_1, x_2) \in [0,1]^2\right\} \tag{9}$$

Recall that (8) must be affiliated and exchangeable.[10] The affililation (or equivalently density log-supermodular) holds if and only if

$$\frac{\partial^2}{\partial x_1 \partial x_2}\sum_{i\in I}\sum_{j\in I}\psi_{i,j}\phi_i(x_1)\phi_j(x_2) \geq 0 \tag{10}$$

for every $(x_1, x_2) \in [0,1] \times [0,1]$, infinitely many constraints. We use normalized B splines to construct $\{\phi\}$, because, then, (10) reduces to a finite number of linear inequality constraints. (See Beresteanu (2007).) In addition, the exchangeability equivalent to

[10] One might want to use a copula to account for a correlation structure. In this case, the nonparametric (Bernstein) copula of Sancetta and Satchell (2004) should be employed, because we want the specification to be flexible. However, there is no convenient way to characterize the density affiliation. Even if there is, the specification would be needlessly complicated, because we need to use another flexible specification for the marginal density separately. Only when we use the Bernstein density of Petrone (1999a,1999b) for the marginal, there is a simplification, which, however, would actually lead back to (8) with a slightly different h than (9).

$\psi_{i,j} = \psi_{j,i}$ for all $i, j \in I$. Let ψ be a vector of $\psi_{i,j}$ with $i \geq j$. Then, there is a matrix A such that (8) is affiliated and exchangeable if and only if

$$A\psi \geq 0 \tag{11}$$

Now, we simply put zero prior on the set of ψ's violating (11) to impose the affiliation. The likelihood simulation procedure is similar to the IPV above. However, the detailed computation algorithm differs, because we need to draw $N(= 2)$ dimensional random vectors and evaluate more complicated bidding function. See Appendix A.2 for normalized B splines and computation algorithm.

2.4 Discussion

For the IPV, Donald and Paarsch (1993,1996), Laffont, Ossard, and Vuong (1995), and Li and Vuong (1997) specify the valuation distribution directly, and then derive the implied likelihood or moments of the bid distribution. These direct methods have been applied only for simple specifications, because in more general cases it is either computationally impractical or difficult to choose the optimal set of moment restrictions.

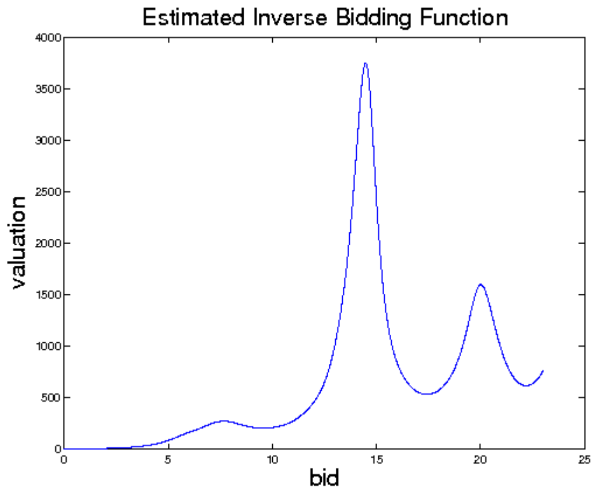
For this reason, Guerre, Perrigne, and Vuong (2000) develop the indirect method for the IPV models. They discover that the inverse of (2) can be expressed as

$$\beta^{-1}(b) = b + \frac{G(b)}{(N-1)g(b)} \tag{12}$$

where G denotes the marginal of the bid distribution and g be its density.[11] This suggests that if we knew G and g correctly, we could trace back the latent valuation $v_{i,t}$ for each observed $b_{i,t}$ in z. Guerre, Perrigne, and Vuong construct $\widehat{v}_{i,t} := b_{i,t} + \frac{\widehat{G}(b_{i,t})}{(N-1)\widehat{g}(b_{i,t})}$ using the empirical CDF $\widehat{G}$ and a kernel estimate $\widehat{g}$. Then, they consistently estimate the valuation density over these pseudo values $\{\widehat{v}_{i,t}\}$ nonparametrically. Since this method is fully flexible and computationally simple, it has been widely used.

[11] Its derivation is simple. The expected utility of bidder i bidding b_i can be expressed by $(v_i - b_i)F(\beta^{-1}(b_i))^{N-1}$. The first order condition is $-F(\beta^{-1}(b_i))^{N-1} + (v_i - b_i)(N-1)F(\beta^{-1}(b_i))^{N-2}f(\beta^{-1}(b_i))/\beta'(\beta^{-1}(b_i)) = 0$. From this, we obain $v_i = b_i + \frac{F(\beta^{-1}(b_i))}{(N-1)f(\beta^{-1}(b_i))/\beta'(\beta^{-1}(b_i))}$. Note that $G(b) = F(\beta^{-1}(b))$ and $g(b) = f(\beta^{-1}(b))/\beta'(\beta^{-1}(b))$.

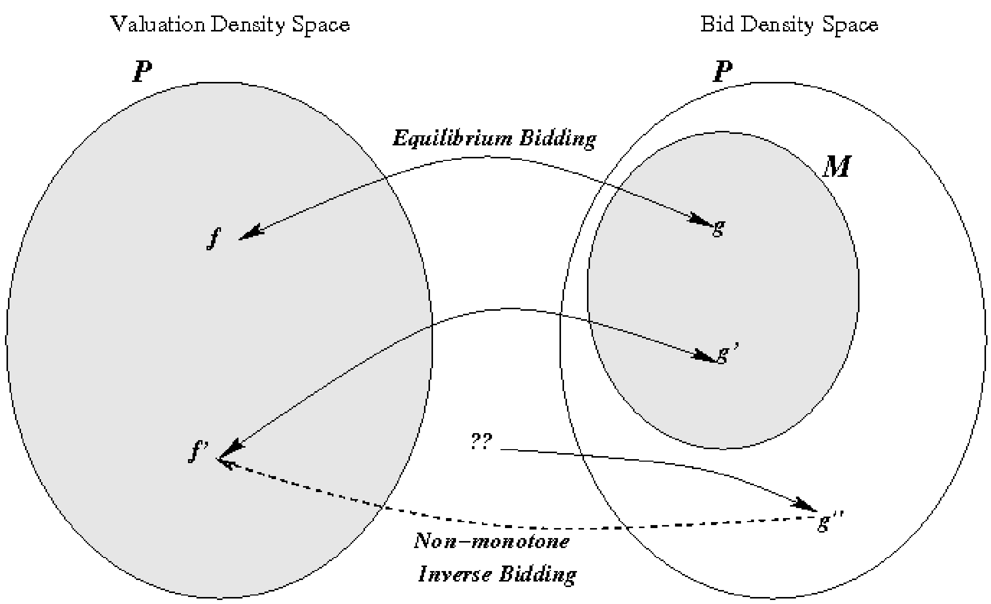
Figure 1: Nonincreasing Inverse Bidding Function for OCS wildcat auctions



However, $\widehat{g}$ is estimated without imposing the monotonicity of (12). If $\widehat{\beta^{-1}}(\cdot)$ is non-increasing, a small bid would be linked with a large value, and vice versa. Then, valuation density estimates based on the 'distorted' values would not be accurate. Figure 1 displays the estimated inverse bidding function for the sample from the OCS wildcat auctions that has been investigated in the empirical literature.

Consider Figure 2 in which P denotes the set of all densities on $\Re_+$. Since values are only need to be independence, every member in P is a potential valuation density, constructing the valuation density space (left side). Let $M \subset P$ be the set of bid densities with a strictly monotone (12). Guerre, Perrigne, and Vuong (2000) show that (2) implies a one to one mapping between M and P. For example, any bid density in M can be seen as an equilibrium outcome rationalized by some valuation density in P, e.g., $g \in M$ and $f \in P$. But, a bid density not in M, say g'', cannot be an equilibrium outcome. Note that though (12) implied by g'' would connect g'' with a density in P, say f', the equilibrium bid density for f' is not g'' but some other bid density, $g' \in M$. Note that similarly, the method of the APV models by Li, Perrigne, and Vuong (2002) does not impose the affiliation, either: the pseudo values would be computed based on the non-affiliated bid density and the valuation density estimate would not be affiliated.

Figure 2: Mapping between Bid Density Space and Valuation Density Space (For the IPV)



The idea of this paper is to rule out all the bid densities that do not meet equilibrium emplications, e.g., M^c for the IPV case. Then, our method would provide a more precise inference and as a result policy implications would be more reliable. However, though very flexible, our analysis is not fully nonparametric like the indirect methods. Hence, the precision improvement of our methodology should be restricted to relatively well-behaved valuation distributions. Even in this case, we sacrifice the computational simplicity to gain the additional precision. Therefore, our methodology complements the indirect approaches.

3 Monte Carlo

We compare our method (BSL) with the method of Guerre, Perrigne, and Vuong (2000) (GPV). For BSL we construct $\{\phi\}$ in (7) using the Legendre polynomials and control the smoothness using a prior. We use $E[f(\cdot|\theta)|y]$ (posterior mean, or predictive density) as the valuation density estimate.[12] For GPV we follow the rule of thumb of Guerre, Perrigne, and Vuong (2000) to choose bandwidths. As appears soon, GPV with the rule of thumb tends to be noisy. Therefore, one might want to use another set of bandwidths to reflect his prior beliefs on smoothness. To take this into account we consider the bandwidths combination minimizing the mean integrated squared error (MISE).[13] That is, we also compare BSL with the precision upper bound of GPV which we call 'Oracle GPV'.[14] For a fixed valuation density, we employ 1,000 Monte Carlo replications. For each replication, we generate a new sample of size $T \times N = 200 \times 2$ (similar to the OCS wildcat data) and run BSL, GPV, and Oracle GPV.

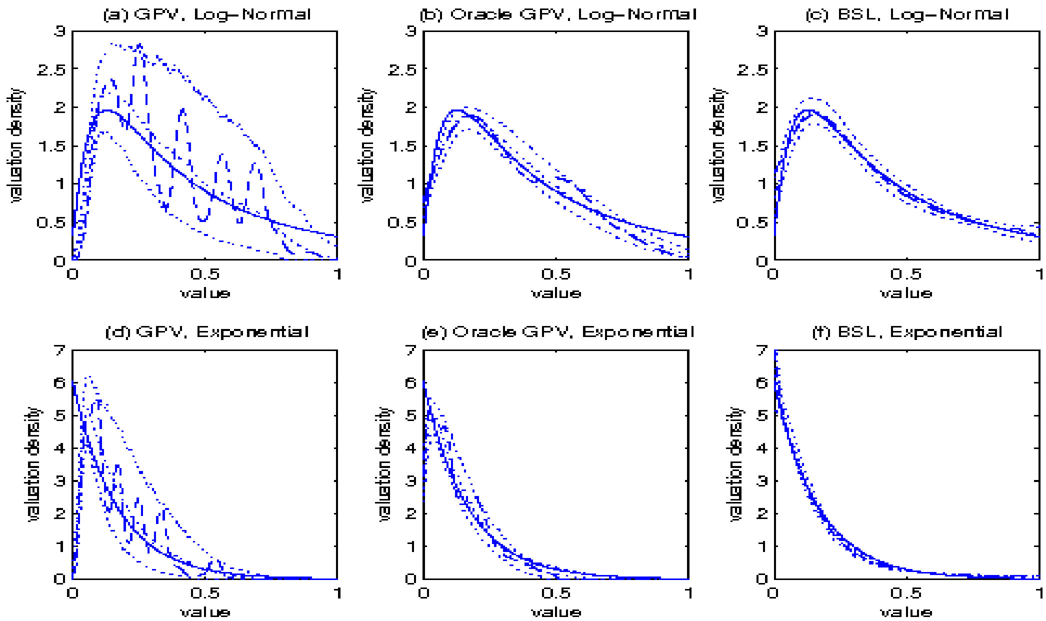
We try four different valuation densities. First, we take the valuation distribution that Guerre, Perrigne, and Vuong (2000) have used for their Monte Carlo study. That is, the truncated log-normal with parameter 0 and 1 with support $[0.055, 2.5]$. We rescale it so that its support is $[0, 1]$. We call it 'Log-Normal'. Second, the truncated exponential distribution with mean 1/6 and support $[0, 1]$, denoted by 'Exponential'. Figure 3 summarizes the results for Log-Normal and Exponential. Each panel plots the true valuation density (plain line), point-wise 5-th, 95-th percentiles, and point-wise mean (dotted) along with a typical density estimate. (from the last Monte Carlo replication) Apparently, BSL provides more precise estimates than GPV. BSL has much narrower 90% frequency bands, closely approximates the true density, and better behaves near the boundaries. Furthermore, Table 1 indicates that BSL is even more precise than Oracle GPV. That is, GPV cannot be better than BSL for these valuation distributions. Table 2 displays the information loss of GPV. Its first column represents the average percentage of observed

[12] Alternatively, we could use $f(\cdot|\widehat{\theta}_B)$ with $\widehat{\theta}_B = E[\theta|y]$. We find this gives only slightly different estimates.

[13] The MISE is a precision measure of the density estimate. Let $\hat{f}_z$ be an estimate for the true density f_0. Then, $MISE(\hat{f}_z) = \int E_z \left(\hat{f}_z(x) - f_0(x)\right)^2 dx$ which is decomposed into $\int V_z \left(\hat{f}_z(x)\right) dx + \int \left(E_z\hat{f}_z(x) - f_0(x)\right)^2 dx$. Thus, the MISE is small, only when both variance and bias are small.

[14] This is infeasible, because we can choose such bandwidths only when we know the true valuation density.

Figure 3: Monte Carlo Results for Log-Normal and Exponential



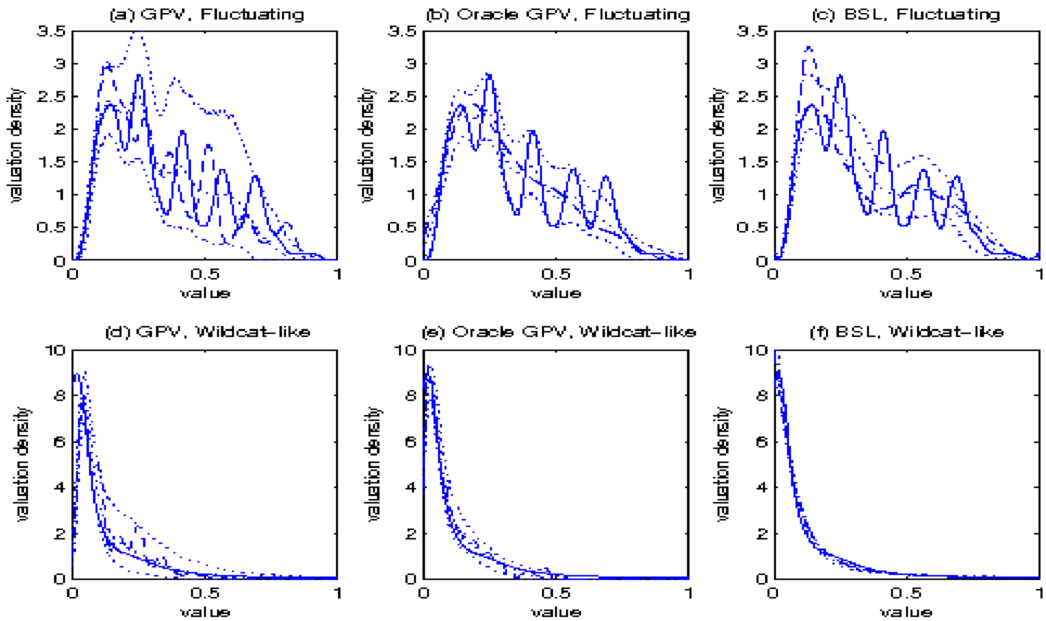
bids trimmed out due to the boundary problem and its second column measures the fraction of sample affected by the monotonicity violation.

However, for Log-Normal and Exponential, BSL is expected to be superior to GPV, because it controls the smoothness and GPV seems to be too flexible. For the third data generating process, we take the typical GPV estimate under Log-Normal (dashed-line on Figure 3(a)), 'Nonsmooth'. In this case, GPV might perform better, because (i) BSL would not be sufficiently flexible, (ii) it uses a prior working against, and (iii) Nonsmooth is actually an outcome from GPV. Figure 4(a)(b)(c) show the Monte Carlo results; BSL is not as accurate as previous experiments, but GPV gets noisier too, producing even much wider frequency band. However, Oracle GPV has lower MISE than BSL (Table 1). This implies that there is a bandwidths combination for GPV to be more precise.

The fourth is the marginal of predictive valuation density estimate of the OCS wildcat auctions (Figure 6(b)).[15] Overall, the Monte Carlo results are very similar to Exponen-

[15] Though this estimate comes from the APV model, we conduct this Monte Carlo under the IPV, because,

Figure 4: Monte Carlo Results for Nonsmooth and Wildcat-like



tial.[16] To make a connection to the next section, we compare the expected revenues for the policy implication on reserve prices under GPV and BSL.[17] We find that BSL is 18.30% more profitable than GPV for Wildcat-like. For other experiments, BSL gains revenue increase by 5.52% for Log-Normal, 8.43% for Exponential, and 3.43% for Nonsmooth.

4 Estimation and Auction Design for OCS Wildcats

We apply our methodology to the OCS wildcat auctions to simulate the posterior of the valuation distribution. We provide the valuation/bid density estimates and choose a reserve price maximizing the seller's future expected revenue using the decision theoretic

otherwise, it would take too long.

[16]Figure 4 second row displays the experiement results. See also Table 1 and Table 2.

[17]In particular, for GPV, we implement the method of Li, Perrigne, and Voung (2003) to compute the revenue maximizing reserve price implied by the estimated bid density. For BSL, we use the Bayesian decision method of Kim (2008) with the posterior from BSL.

Table 1: MISE Comparisons

DGP's	GPV	Oracle GPV	BSL	$\frac{\text{BSL}}{\text{GPV}} \times 100\%$
Log-Normal	0.23805	0.02955	0.01034	4.344
Exponential	0.95726	0.15037	0.02430	2.539
Nonsmooth	0.29562	0.14500	0.16283	55.080
Wildcat-like	1.04260	0.20158	0.06406	6.144

Table 2: Information Loss of GPV (%)

DGP's	Triming Rate (boundaries)	Distortion Rate (Monotonicity Violation)	Total
Log-Normal	10.918	13.993	24.911
Exponential	18.420	15.503	33.923
Nonsmooth	7.567	14.772	22.339
Wildcat-like	18.883	14.404	33.288

method introduced by Kim (2008).

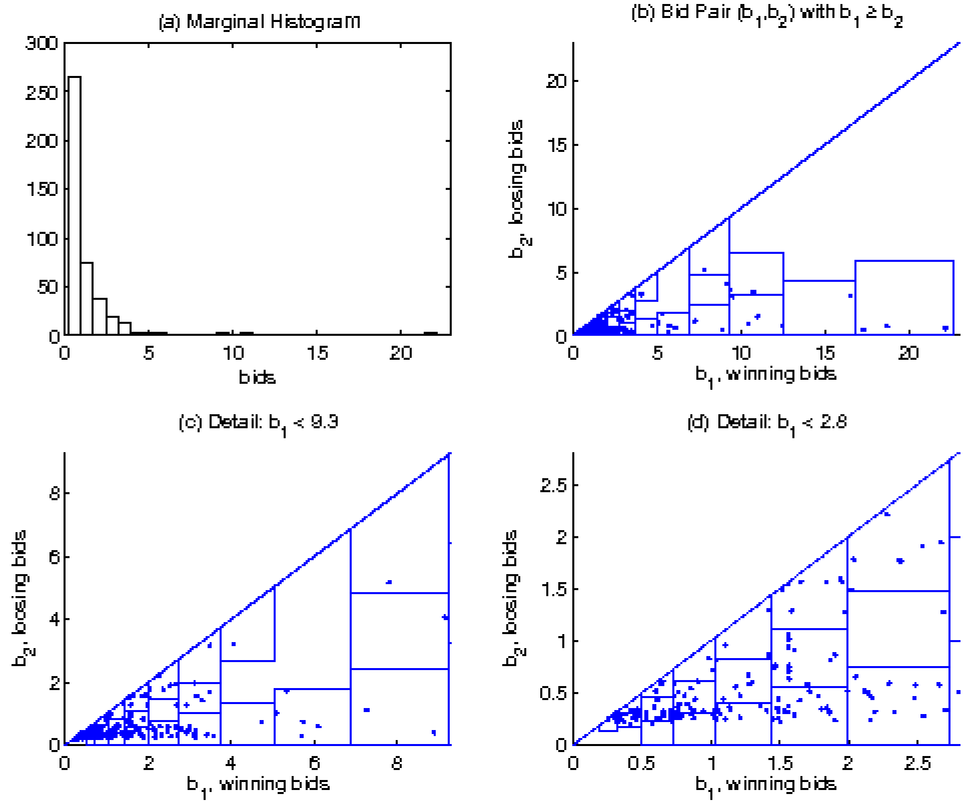
4.1 Data and Sample Space Discretization

We discuss the OCS wildcat data briefly.[18] The sample consists of 217 auctions, each having two bids in 1972 dollars. It contains the auctions between 1954 and 1970 among the sales held by the U.S. federal government to sell its mineral right on oil and gas on offshore lands off the Texas and Louisiana coasts.

Figure 5(a) is the marginal histogram of the sample, which roughly resembles an exponential density. The sample mean and standard deviation are 1.458 and 2.557 ($\times$100 dollars for each). Panel (b) shows how the bid pairs (b_1, b_2) with $b_1 > b_2$ are scattered. Most bids are condensed around the origin, while the sample has a long tail.

[18] For a thorough data description, see McAfee and Vincent (1992), Hendricks and Porter (1992), Hendricks, Pinkse, and Porter (2003), and Li, Perrigne, and Vuong (2000,2003). The dataset is publically available at `http://capcp.psu.edu/index.html`

Figure 5: OCS Wildcat Data and Bid Space Discretization



Though we observe a few auctions whose b_1 is much larger than b_2, the overall sample correlation is 0.412. To examine closely, panel (c) and (d) enlarge the parts for $b_1 < 9.3$ and $b_1 < 2.8$, respectively.

We maintain the assumptions that Li, Perrigne, and Vuong (2000, 2003) rely on for this sample: nonbinding reserve price, no dynamic consideration, the auction homogeneity, and the symmetric APV paradigm. Note that they explain the reason that these assumptions are reasonable for the dataset.[19] Though most of these assumptions have

[19]For example, they claim that the actual reserve price $15 is too low to be an effective screening device and the game induced by the auction mechanism can be seen as symmetric, because all bidders have equal

been taken by other researchers who use the same sample, there has been a disagreement on the APV hypothesis. For example, Li, Perrigne, and Vuong (2000, 2003) and Campo, Perrigne, and Vuong (2003) take the APV paradigm, while Hendricks, Pinske, and Porter (2003) support the pure common value (PCV) paradigm. However, they all agree that the reality would be better explained by the affiliated value (AV) paradigm that embraces both APV and PCV as special cases.[20]

Last, to obtain an unbiased simulated likelihood, we discretize the sample space as indicated in Figure 5. In panel (b) (and (c) and (d), too), when two adjacent squared bins on the horizental axis, the right one is 80% larger than the left one. Then, for each of column, we construct additional bins by putting equally spaced squares until a trapezoid that cannot contain another squared bin remains. We constructed 30 nonempty bins.

4.2 Valuation/Bid Density Estimation

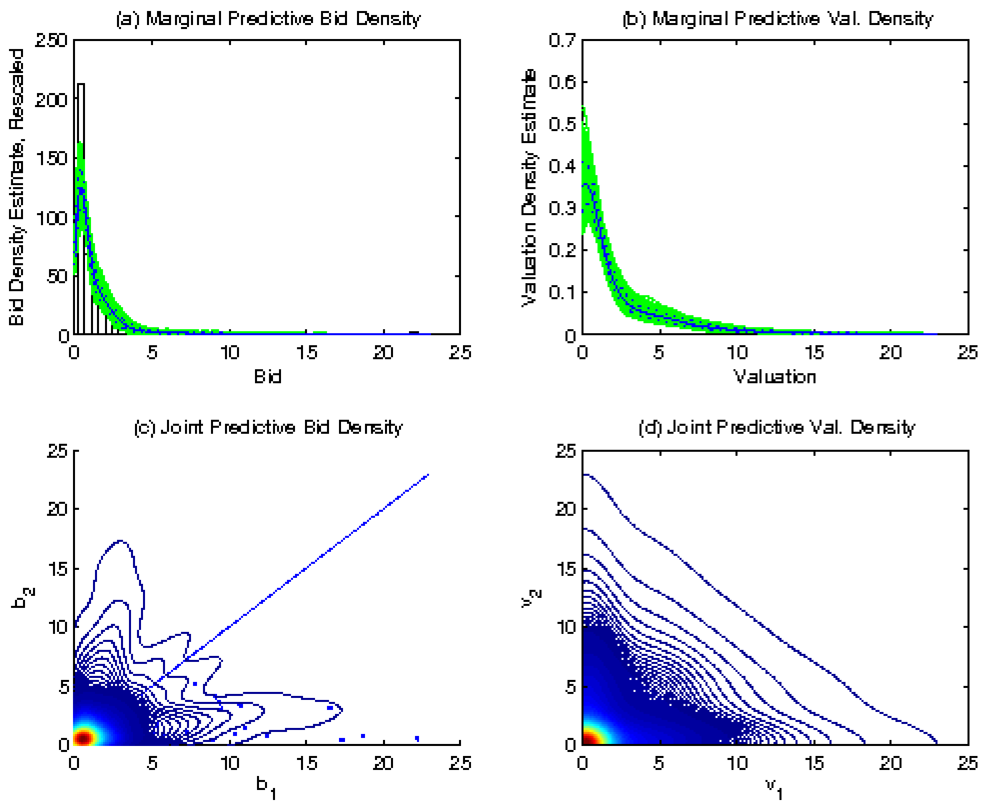
We model the valuation density using (8) with (9). In particular, to impose the affiliation restriction, we employ the normalized B splines to construct $\{\phi_i\}$. Since the bids are exponential-like distributed, we choose an exponential with hazard rate μ for $\tilde{f}(\cdot|\mu)$ in (8). We employ ψ with 91 components. This is sufficiently flexible and computationally practical. We use a prior to constrol the tail behavior. As we discuss later in 4.4, we discover the discontinuity of the optimal reserve price in tail behavior of bid distribution. Thus, even if a bid density estimate closely approximates the true one, the optimal reserve price associated with such a 'good' estimate can be unreasonably different from the true price. The Adaptive Metropolis algorithm of Haario, Saksman, and Tamminen (2001) simulates the posterior. Appendix E provides all the details for the specification and implementation.

To obtain a valuation density function that preserves the affiliation property, we compute the posterior mean of log density, i.e., $E[\log f(v_1, v_2|\theta)|y]$. Then, we use exponential of this as our valuation density estimation. This posterior mean is consistently estimated by the AM output. Recall that we aim to estimate the valuation density first

opportunity to access the same information on the auctioned tract.

[20] The AV paradigm is introduced by Wilson (1977).

Figure 6: Predictive Densities for OCS wildcat auctions



and also to provide a policy recommendation for auction desigin. Since we conduct policy analysis using the posterior of the valuation density, the bid density estimate do not have any importance for our purpose. However, in order to check the goodness-of-fit, it may be useful to construct a bid density estimate, though this is not our main purpose. Recall that it is computationally hard to derive the theoretical bid density, we approximate it using simulation. That is, we generate many auctions under θ_s for each $s = 1, \ldots, S$, and estimate the implied bid density using a kernel method over these simulated bids.

We summarize the predictive estimation results in Figure 6. Panel (a) plots the posterior distribution of the marginal bid density $\widehat{g}(b|\theta_s)$ with the AM output $\{\theta_s\}_{s=1}^{S}$. The solid line is the predictive marginal bid density and the dashed lines construct a 90% credible band. This predictive bid density estimate explains the actual data fairly well and its narrow credible band implies the accuracy of inference. But, it does not closely fit the tail part, because we control the tail behavior and use large bins so that the prior is more informative than the sample points there. However, as appears on panel (c), the joint predictive density explains some negative correlation pattern over the tail area. Similarly, panel (b) plots the posterior distribution of marginal valuation densities $f(v|\theta_s)$. Since a bidder bids less than his valuation, the valuation densities spread out toward large values. The predictive marginal valuation density is represented by the plain line. This resembles an exponential density, though it is a bit more covex to the origin. The credible band is also narrow, which implies that the posterior is very condensed around its mean. Panel (d) is the 1,000 level curve contour of joint predictive valuation density. The plug-in estimation results are almost identical to Figure 6: we do not report them, here.

4.3 Auction Design with Bayes Rule of Kim (2008)

Though the valuation density estimation is interesting by its own, more important feature of structural analysis of bid data would be its ability to provide a policy implication on auction design. Riley and Samuelson (1981) show that, under the IPV paradigm, the revenue maximizing reserve price is given by $\rho_R(\theta) := \rho^*$, which solves $\rho^* = v_0 + \frac{1-F(\rho^*|\theta)}{f(\rho^*|\theta)}$ when the seller's valuation is v_0. Paarsch (1997) proposes $\rho_R(\widehat{\theta}_{ML})$ as a consistent estimator with the maximum likelihood estimator $\widehat{\theta}_{ML}$. Later, Li, Perrign, Vuong (2003) extend this idea to the APV paradigm and apply their method to the OCS wildcat auctions. Especially, they observe that the revenue maximizing reserve price can be expressed as a functional of the bid distribution. Hence, they nonparametrically estimate the bid density first and then choose the revenue maximizing price with respect to the estimated bid distribution. We consider these decision procedures as a plug-in rule, because they substitute the estimate for the true quantity.

Generally, a plug-in rule does not produce the highest payoff. It pins down the maximizer for the 'estimated' payoff, but does not take into account the overall shape of the payoffs around, possibly including, the true one. This can be important when the state of nature is not certainly known. To see the reason, suppose that the true payoff is gradually increasing up to its maximum and sharply decreasing thereafter. Then, if the seller knows this structure, he must prefer underestimation of the maximizer to overestimation for an equal size of error. Note that the size of sampling error is closely related to parameter uncertainty. Therefore, we can obtain a higher payoff by incorporating the payoff structure and parameter uncertainty into the decision procedure.

For this purpose, Kim (2008) introduces a Bayesian decision theoretic framework. Let $\Pi(\theta, \rho)$ denote the seller's payoff (expected revenue) for the reserve price ρ under θ and $\mathcal{A}$ be the set of all feasible reserve prices. The Bayes rule selects a Bayes action defined by

$$\rho_B(y) := \arg\max_{\rho \in \mathcal{A}} \int \Pi(\theta, \rho) p(\theta | y) d\theta \tag{13}$$

Observe that the posterior systematically quantifies parameter uncertainty and the Bayes rule considers the average payoff structure that is weighted by the posterior. Kim (2008) discusses the optimality of (13) as a decision rule. We revisit the auction design problem for the OCS wildcat auctions using the Bayes rule.

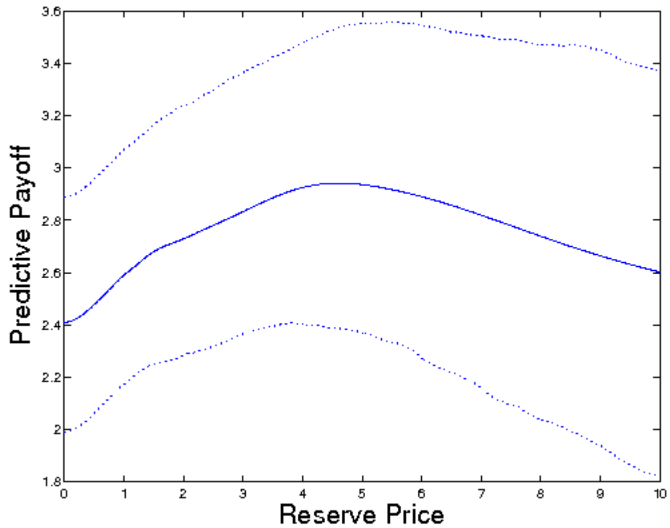
We take $\mathcal{A} := \{0, 0.01, \ldots, 9.99, 10\}$ as the set of feasible reserve prices, since the valuations greater than 10 would have a very small density as Figure 6 suggests. For simplicity, we assume the seller's valuation is zero: $v_0 = 0$. Then, the seller's payoff can be written as

$$\Pi(\theta, \rho) = E\left[\beta(v_{(2)} | \rho, \theta) \cdot 1\left(v_{(2)} > \rho\right) \middle| \theta\right] \tag{14}$$

where $v_{(1)} < v_{(2)}$. For a given θ, we consistently estimate (14) using Monte Carlo for each $\rho \in \mathcal{A}$. Let $\widehat{\Pi}(\theta, \rho)$ denote this estimate. Then, we approximate (13) using the MCMC output.

We find $\widehat{\rho}_B(y) = 4.62$. Thus, the reserve price \$462 per acre is decision theoretically optimal. We also computed the predictive revenue of this reserve price: $\int \Pi(\theta, \widehat{\rho}_B(y)) p(\theta | y) d\theta = 294.24$. This implies that, since the most wildcat tracts are 5,000 acres, the seller's

Figure 7: Predictive Revenue for OCS wildcat auctions



predictive revenue is \$1,471,197 ($= 294.24 \times 5,000$). This is much higher than the predictive revenue of the actual reserve price of \$15, which is computed as \$1,208,783. Thus, our counterfactual analysis indicates that if our proposal is adopted, the U.S. government would increase the revenue by \$262,414. When we take into account the fact that hundreds tracts have been offered annually, this revenue gain is significant.

To make a comparison, we compute $\rho_R(\widehat{\theta}_B) = 4.47$ whose corresponding future revenue is \$1,470,872. Therefore, we gain \$325 additionally by considering the payoff structure and parameter uncertainty. However, this is small. It might be either because the payoff structure is symmetric or because the amount of parameter uncertainty is small. In our case, the former is the main reason as appears in Figure 7. The predictive payoff (plain line) is plotted along with a 90% credible band (dashed lines). The wide credible band suggests that there is a fairly large amount of parameter uncertainty, but the predictive revenue curve is roughly symmetric about its maximum. Note that the predictive revenue for zero reserve price is very close to the sample mean of winning bids, 2.243, indicating the accuracy of our analysis.

Note that our choices are very different from the estiamte \$273 of Li, Perrigne, and Vuong (2003). Hence, this difference can be explained by the fact that we exploit the additional information: monotonicity of equilibrium bidding, affiliation of valuations, and prior beliefs on the tail behavior. Since the sample is small, the contribution of the additional equilibrium information plays an important role.

Finally, many researchers have reported that the actual price of \$15 is too low to be an effective screening device. However, the U.S government has not revised the auction format. Our results strengthen the opinion of the economist by proposing an even higher reserve price as an optimal decision.

4.4 Note on discontinuity of $\rho_R(\cdot)$

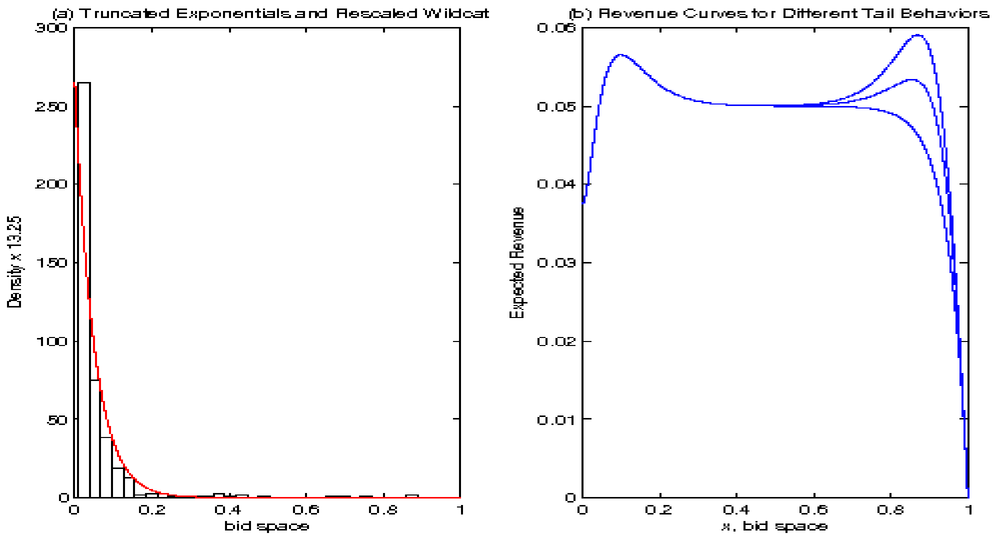
For a simple presentation, consider an IPV auction with two bidders. Li, Perrigne, and Vuong (2003) show that when seller's valuation is zero, the expected revenue is given by

$$\tilde{\Pi}(x) := 2E\left[\frac{G(x)^2}{g(x)G(b_1)} \cdot 1\{b_1 > b_2\}\, 1\{b_1 > x\}\right] \tag{15}$$

for a point x and that for x_0 maximizing (15) the revenue maximizing reserve price is given by $\rho^* = \beta^{-1}(x_0) := x_0 + G(x_0)/g(x_0)$. Now, suppose that the equilibrium bids have a density $g(b|c) \propto \left(1 - 0.05e^{-0.05b} + c\right) \cdot 1\{b \in [0,1]\}$ with $c > 0$, tail thickness. Figure 8 (a) plots three almost identical densities with different thickness $(c_1, c_2, c_3) = (0, 5 \cdot 10^{-8}, 10^{-7})$. Panel (b) displays the corresponding revenue curves, (15). They are very similar for $x < 0.6$ with a local maximum at $x_0 = 0.1$, but differ for $x > 0.6$: the second maximum grows as the tail gets thicker. in particular, the global maximum shifts from $x_0 = 0.1$ to $x_1 = 0.8703$. When we evaluate the associated reserve prices, $\rho^*(c_1) \approx \rho^*(c_2) \approx 0.4195$, but, strikingly, $\rho^*(c_3) \approx 1,534,500$! Thus, if the tail behavior is not properly controlled, an unacceptable policy suggestion can arise.

Our dicussion here is relevant to the wildcat auctions since these densities explain the data histogram fairly well. In fact, when we analyze the wildcat data without considering this issue, the estimated payoff functions often have two local maxima, one related to

Figure 8: Discontinuity of Revenue Maximizing Reserve Price



a reasonable reserve price, the other, which is usually the global, to a very large price such as several millions.

5 Summary and Extensions

We directly model the valuation distribution and simulate the likelihoods. When the simulation is unbiased, the posterior is exact even with a small simulation size. Our method is flexible and computationally practical. Moreover, since we impose all the available information (theoretical restrictions and prior beliefs), our inference may be more precise than the indirect approaches, as shown in the Monte Carlo studies in section 3. In addition, the Bayesian approach provide a natural environment for a formal decision framework. In particular, we choose a reserve price for the OCS wildcat auctions using the Bayes rule of Kim (2008). We formally consider the payoff structure and parameter uncertainty and select an action through the optimal decision method. Our choice is quite different from the estimate of Li, Perrigne, and Vuong (2003).

We have developed our framework under restrictive assumptions such as identical valuation distribution across auctions, zero reserve price, fixed N (number of potential bidders), etc. However, our method can easily extend to more general situations. For example, suppose the bidders' valuation depend on auction characteristics denoted by X. Then, we may model θ further so that it depends on X under some structural assumptions. For a simple example, we may let $\theta(X) := (\mu(X), \psi)$ with $\mu(X) = X\gamma$. Then, we put prior on (γ, ψ) and the rest of anlaysis would be similar. Second, suppose $\rho > 0$. Then, the number of actual bidders n may vary, though N is fixed. Therefore, the sample space is now two dimensional, (n, b) and there is an obvious sample space discretization, which would construct the likelihood $L(\theta|y) := \prod_n \prod_d \{\pi_{n,d}(\theta)\}^{y_{n,d}}$.

We observe many interesting features from the OCS auctions that might alter the policy implications when considered formally. The sample that we, and previous researchers, have investigated is only small fraction of the entire OCS wildcat auctions. Many auctions have more than or less than just two bidders. Relatedly, the U.S. govenmnet offers at least one hundred tracts whenever it holds an auction. This implies that the potential bidders would strategically choose the tracts for which they seriously bid, perhaps conditional on the areas in which they are already operating. In addition, as we discussed earlier, some other researchers have adopted the PCV. Thus, a policymaker would be interested in having knowledge about the risk of using a 'wrong' model when he makes a decision. Therefore, modelling a decision tool incorporating all of these issues would be important and we plan to address them in separated projects.

Appendix

A Simulation Algorithm

A.1 Sampling Valuations and Bidding Function Evaluation for IPV

The auction simulation algorithm consists of two steps as follows. H denotes the CDF of h and $x_{(r)}$ is the r-th smallest out of $x_1, x_2, \ldots$. Under given θ,

1. Draw a random sample of size R, $\tilde{v}_1, \ldots, \tilde{v}_R$.
   Let $\{\tilde{u}_r\}_{r=1}^{R}$ be uniform draws. Then, $\tilde{v}_r := F^{-1}(\tilde{u}_r|\theta) = \tilde{F}^{-1}(H^{-1}(\tilde{u}_r|\psi)|\mu)$. To approximate $H^{-1}(\tilde{u}_r|\psi)$, we evaluate $H(\cdot|\psi)$ on each point in $\{0, 0.01, \ldots, 0.99, 1\}$ and apply a monotonicity preserving interpolation.[21] Most statistics softwares evaluate $\tilde{F}^{-1}(\cdot|\mu)$.

2. Compute the equilibrium bids $\tilde{b}_1, \ldots, \tilde{b}_R$ using (2).
   Note that $F(\tilde{v}_r|\theta) = \tilde{u}_r$. Hence, a trapezoid rule over $\left((0,0), \left\{\left(\tilde{v}_{(r)}, \tilde{u}_{(r)}^{N-1}\right)\right\}_{r=1}^{R}\right)$ solves $\int_0^{\tilde{v}_r} F(\alpha|\theta)^{N-1} d\alpha$. That is,
   $$\int_0^{\tilde{v}_{(r)}} F(\alpha|\theta)^{N-1} d\alpha \approx \int_0^{\tilde{v}_{(r-1)}} F(\alpha|\theta)^{N-1} d\alpha + \frac{1}{2}\left(\tilde{u}_{(r-1)}^{N-1} + \tilde{u}_{(r)}^{N-1}\right)\left(\tilde{v}_{(r)} - \tilde{v}_{(r-1)}\right)$$
   starting from $\int_0^{\tilde{v}_{(1)}} F(\alpha|\theta)^{N-1} d\alpha \approx \frac{1}{2}\tilde{u}_{(1)}^{N-1}\tilde{v}_{(1)}$.

This trapezoid is fairly accurate. For the part where F quickly increases, we may have many reference points because they are random sample from F. For the part where F is almost flat, we may not have many reference points, because the probability density is small. However, the area to be integrated is almost rectangle.

A.2 Sampling for APV: $(\tilde{v}_{1,1}, \tilde{v}_{2,1}), \ldots, (\tilde{v}_{1,R}, \tilde{v}_{2,R}) \sim f(\cdot|\theta)$

We employ an accept/reject method.[22] In particular, we use a piecewise uniform mimicking (9) for the source function. That is, we use the kernel of (9) given by $k_h(x_1, x_2|\psi) :=$

[21] For example, a piecewise linear function interpolation or the piecewise cubic interpolation method of Fritsch and Carlson (1980).

[22] The inverse CDF not only runs slowly in a multivariate case but also gives no help to computing (1) as in the IPV.

$\exp\left\{\sum_{i\in I}\sum_{j\in I}\psi_{i,j}\phi_i(x_1)\phi_j(x_2)\right\}$. Then, construct the kernel for the piecewise uniform as follows

$$k_{pu}(x_1,x_2|\psi)=k_h\left(\frac{i}{10},\frac{j}{10}\middle|\psi\right)+k_h\left(\frac{i}{10},\frac{j+1}{10}\middle|\psi\right)+k_h\left(\frac{i+1}{10},\frac{j}{10}\middle|\psi\right)+k_h\left(\frac{i+1}{10},\frac{j+1}{10}\middle|\psi\right)$$

for $(x_1,x_2)\in[\frac{i}{10},\frac{i+1}{10}]\times[\frac{j}{10},\frac{j+1}{10}]$ $i,j\in\{0,1,2,\cdots,9\}$ Then, the accept/reject algorithm is as follows. For $r=1,\cdots,R$.

1. Draw a proposal $(\tilde{u}_1^*,\tilde{u}_2^*)$ from the density proportional to k_{pu}
2. Let $(\tilde{u}_{1,r},\tilde{u}_{2,r})=(\tilde{u}_1^*,\tilde{u}_2^*)$ with probability $\frac{k_h(\tilde{u}_1^*,\tilde{u}_2^*|\psi)}{Qk_{pu}(\tilde{u}_1^*,\tilde{u}_2^*|\psi)}$ with $Q\geq\sup_{(u_1,u_2)\in[0,1]^2}\frac{k_h(u_1,u_2|\psi)}{k_{pu}(u_1,u_2|\psi)}$
3. If the proposal is not accepted, go back to step 1.

Once $(\tilde{u}_{1,1},\tilde{u}_{2,1}),\cdots,(\tilde{u}_{1,R},\tilde{u}_{2,R})$ are drawn, then $(\tilde{v}_{1,1},\tilde{v}_{2,1}),\cdots,(\tilde{v}_{1,R},\tilde{v}_{2,R})$ are obtained by $\tilde{v}_{i,r}=\tilde{F}^{-1}(\tilde{u}_{i,r}|\mu)$ for each $i=1,2$ and $r=1,\cdots,R$.

A.3 Evaluation of Equilibrium Bidding Function for APV

We compute the equilibrium bids $\left\{(\tilde{b}_{1,r},\tilde{b}_{2,r})\right\}_{r=1}^{R}$ using (1). It may be quite time consuming to evaluate (1), since it requires us to estimate many triple integrals. However, we find that the following recursive algorithm reduces the computing time, significantly, which has three steps.

1. We compute $a_1,\cdots,a_{2R}$ with

$$a_j:=\frac{f(\tilde{v}_{(j)},\tilde{v}_{(j)}|\theta)}{\int_0^{\tilde{v}_{(j)}}f(\tilde{v}_{(j)},t|\theta)dt}$$

   for $j=1,\cdots,2R$ where $\tilde{v}_{(j)}$ is the j-th smallest out of the $R\times 2$ simulated random values. Note that the integral on the denominator is quickly estimated by the Gaussian quadrature. [23]

[23] The upper limit of the integral can be a very large number and, in that case, the Gaussian quadrature approximation with finite points may be poor. However, the integral can be expressed as $\int_0^{\tilde{F}(\tilde{v}_{(j)}|\mu)}h(\tilde{F}(\tilde{v}_{(j)}|\mu),s|\psi)ds$ for which the upper limit is always less than one. Then, for this integral, we find that the Gaussian quadrature even with three points provides fairly accurate estimates. Moreover, $\tilde{F}$ should not be evaluated, because we already have $\tilde{u}_{i,r}=\tilde{F}(\tilde{v}_{i,r}|\mu)$ $i=1,2$, $r=1,\cdots,R$.

2. Constuct

$$A_{i,j} := \int_{\tilde{v}_{(i)}}^{\tilde{v}_{(j)}} \frac{f(u,u|\theta)}{\int_0^u f(u,t|\theta)dt} du \approx A_{i,j-1} + \frac{1}{2}\left(a_j + a_{j-1}\right)\left(\tilde{v}_{(j)} - \tilde{v}_{(j-1)}\right)$$

for $i = 1, \cdots, 2R$ and $j = i+1, \cdots, 2R$ with $A_{i,i+1} \approx \frac{1}{2}(a_i + a_{i+1})(\tilde{v}_{(i+1)} - \tilde{v}_{(i)})$.

3. $L(\alpha|v) := \exp\left\{-\int_\alpha^v \frac{f_{y_1|v_1}(u|u)}{\int_0^u f_{y_1|v_1}(t|u)dt} du\right\}$ is approximated by

$$\int_0^{\tilde{v}_{(j)}} L\left(\alpha|\tilde{v}_{(j)};\theta\right) d\alpha \approx \frac{1}{2}\sum_{i=1}^{j-1}(\exp\left(-A_{i,j}\right) + \exp\left(-A_{i+1,j}\right))(\tilde{v}_{i+1} - \tilde{v}_i)$$

The last two steps amount to a trapezoid rule approximation with reference points $\tilde{v}_{(1)}, \cdots, \tilde{v}_{(2R)}$.

A.4 Alternative Method to Estimate Likelihoods

This bidding function evaluation still takes long. We use an alternative to estimate $\hat{\pi}_d(\theta)$ on the implied valuation space discretization. First, approximate $\beta^{-1}(\cdot|\theta)$ using a monotonicity preserving interpolation: evaluate (1) at every percentile of $\{\tilde{v}_{(1)}, \ldots, \tilde{v}_{(2R)}\}$ and find a strictly increasing function connecting $\left\{\left(\tilde{b}_{p\text{-th }\%}, \tilde{v}_{p\text{-th }\%}\right)\right\}_{p=1}^{100}$. Second, construct the valuation space discretization by evaluating this function at each corner point of the bins in the sample space. Then, estimate $\hat{\pi}_d(\theta)$ by counting the simulated valuations in the d-th bin in the valuation space. Since this procedure evaluates (1) only 100 times regardless of R, we may use a large R.

B Basis Functions and Prior Specification

B.1 Legendre Polynomials and Smoothness Control

We employ the Lengendre polynomials, i.e., $\phi_j(u) := \sqrt{2j+1}\tilde{\phi}_j(2u-1)$ with $\tilde{\phi}_j(x) = \frac{d^j}{dx^j}(x^2-1)^j/(2^j j!)$. Note that ϕ_j gets more noisy as j increases. Hence, to control the smoothness, we use the prior such that $\psi_j \sim N(0, \tau/2^j)$ for $\tau > 0$. Then, ψ_j is more condensed around zero for higher j and the posterior picks a smooth density more likely. We use Unif[0,1] for $\tilde{F}$ just for simplicity.[24]

[24] Hence, the properly selected $\tilde{F}$ may result in smaller MISE's than presented below.

B.2 Normalized B splines

We construct the basis functions as follows:

$$\left\{\phi_i(x) := K\left(\frac{x - i/k}{1/k}\right)\right\}_{i \in I} \tag{16}$$

where $K(x) := \sum_{j=0}^{l} \frac{(-1)^j}{(l-1)!}\left(\frac{l!}{j!(l-j)!}\right)\left(x + \frac{l}{2} - j\right)_+^{l-1} 1\{|x| < l/2\}$ with an integer $l > 1$ and $x_+^a = x^a 1\{x > 0\}$. K is a kernel symmetric about zero and $l-1$ times differentiable with supprot $[-\frac{l}{2}, \frac{l}{2}]$. In addition, I is the set of all integers in $[-m, k+m]$, with an integer $k > l$ and $m := (\text{floor}(l/2) - 1\{l \text{ is even}\})$. ($|I| = 2m + k + 1$, cardinality of I.) Then, $\{\phi\}$ are centered on equidistant grid points $\left\{\frac{-m}{k}, \frac{-m+1}{k}, \ldots, \frac{k+m}{k}\right\}$. Note that (16) are $l-2$ times differentiable and the $l-1$-th derivative does not exist at the grid point. Some are located outside $[0,1]$, since every point in $[0,1]$ must have equal number of nonzero bases for the affiliation condition below.[25]

B.3 Affiliation Restrictions with Normalized B splines

Affiliation is equivalent to density log-supermodular. $\kappa_\psi(x_1, x_2) := \sum_{i \in I}\sum_{j \in I} \psi_{i,j}\phi_i(x_1)\phi_j(x_2)$ for simplicity. Then, (8) is log-supermodular if and only if $\frac{\partial^2}{\partial x_1 \partial x_2}\kappa_\psi(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in [0,1] \times [0,1]$, infinitely many restrictions. However, Beresteanu (2007), using (16), characterizes it by

$$\kappa_\psi\left(\frac{i}{k}, \frac{j}{k}\right) + \kappa_\psi\left(\frac{i+1}{k}, \frac{j+1}{k}\right) \geq \kappa_\psi\left(\frac{i+1}{k}, \frac{j}{k}\right) + \kappa_\psi\left(\frac{i}{k}, \frac{j+1}{k}\right)$$

for all $i, j \in \{0, 1, \cdots, k\}$, which is now k^2 linear inequalities. Furthermore, the exchangeability reduces this to $\frac{k(k+1)}{2}$, because only the ones associated with grid points below (or above) the 45 degree line are relevant due to $\kappa_\psi(x_1, x_2) = \kappa_\psi(x_2, x_1)$. Recall that ψ denotes the vector of $\psi_{i,j}$ with $i \geq j$. Hence, it is $\frac{|I|(|I|+1)}{2}$ dimensional. Therefore, A in (11) is $\frac{k(k+1)}{2} \times \frac{|I|(|I|+1)}{2}$.

[25] For example, ϕ_{-m} is the most left located with a positive tail at zero. That is, $\phi_{-m-1}(0) = 0$, if $-m-1 \in I$. Similarly, $\phi_{k+m}(1) > 0$, while $\phi_{k+m+1}(1) = 0$.

B.4 Tail Behavior

The discontinuity of optimal reserve price with respect to the tail behavior leads us to construct a prior to control the tail behavior. First of all, we assume $p(\mu, \psi) = p(\mu)p(\psi)$ for simplicity. Let $N(a, b)$ denote a normal density with mean vector a and covariance b. For the AM algorithm to converge, the posterior must have a bounded support. Hence, we use a prior whose supports are given by $\mu \in (0, 30]$ and $\max(|\psi|) < 30$. In addition, the prior should be zero for ψ violating (11). Then, we employ the prior in the form of

$$\begin{aligned} p(\mu) &\propto N(25, 1) \cdot 1\{\mu \in (0, 30]\} \\ p(\psi) &\propto N(0, \Sigma_\psi) \cdot 1\{A\psi \geq 0 \text{ and } \max(|\psi|) < 30\} \end{aligned}$$

Now, recall that $\phi_j(\cdot)$ given in (16) is centered around $\frac{j}{k}$. Hence, $\psi_{i,j}$ is related with the shape of (9) around $\left(\frac{i}{k}, \frac{j}{k}\right)$. Let $d_{i,j} := 10 \cdot \max\left\{\left((\frac{i}{k})^2 + (\frac{i}{k})^2\right)^{\frac{3}{2}}, 0.02\right\}$, which is increasing in the distance of $\left(\frac{i}{k}, \frac{j}{k}\right)$ from the origin. We employ a diagonal matrix Σ_ψ whose element associated with $\psi_{i,j}$ equals $d_{i,j}^{-1}$. Then, this prior puts a smaller variance on $\psi_{i,j}$ located further from the origin so that the posterior selects a valuation density whose tail resembles the exponential $\tilde{f}$.

C The Adaptive Metropolis Algorithm

For a given prior $p(\theta)$, we can simulate the posterior using a Metropolis-Hastings algorithm. However, since choosing a good proposal density is hard for a high dimensional θ, we use an adaptive MCMC algorithm. in particular, we employ the Adaptive Metropolis (AM) algorithm of Haario, Saksman, and Tamminen (2001). Suppose θ is d dimensional. Let I_d be the $d \times d$ identity matrix and $s_d := (2.38)^2/d$. At each s-th iteration, we draw a proposal $\tilde{\theta}$ from $N(\theta_{s-1}, \Omega_{s-1})$ where, for a small number $\varepsilon > 0$ and a prespecified initial periods s_0,

$$\Omega_{s-1} := \begin{cases} \Omega_0 & \text{if } s \leq s_0 \\ s_d \widehat{\text{Cov}}(\theta_0, \theta_1, \cdots, \theta_{s-1}) + s_d \varepsilon I_d & \text{otherwise} \end{cases}$$

with Ω_0 and $\widehat{\text{Cov}}$ denoting some initial covariance and the sample covariance, respectively. Then, $\theta_s := \tilde{\theta} \cdot 1\{u < \alpha\} + \theta_{s-1} \cdot 1\{u > \alpha\}$ with $u \sim \text{Unif}[0,1]$ and

$$\alpha := \min\left\{\frac{p(\tilde{\theta})\hat{L}(\tilde{\theta}|y)}{p(\theta_{s-1})\hat{L}(\theta_{s-1}|y)}, 1\right\} \tag{17}$$

Haario, Saksman, and Tamminen (2001) show that the AM algorithm converges to the correct posterior for any θ_0 with $p(\theta_0) > 0$ provided that the posterior is bounded from above and has a bounded support. They also note that we can update Ω_s for any increasing subset of $\{\theta_s\}$

Sampling ψ under $A\psi > 0$

Hajivassiliou and McFadden (1990) and Keane (1990) develop a method, which is called the GHK sampler, to draw a random vector from a truncated normal distribution. Then, Geweke (1995) draws a random vector under a set of linear inequality restrictions using the GHK sampler. As noted earlier, we draw a candidate under the affiliation restrictions (11) in order to obtain a proposal function that generates an acceptable candidate more frequently. For this purpose, we use Geweke (1995) as follows. Let $\bar{A}$ denote a $\frac{|\mathcal{I}|(|\mathcal{I}|+1)}{2}$ dimensional invertable matrix that contains A in (11) as its first $\frac{k(k+1)}{2}$ rows. Then, (11) can be written as

$$\bar{A}\psi \geq \bar{a} \tag{18}$$

for which first $\frac{k(k+1)}{2}$ elements of $\bar{a}$ are zeros and the others are all $-\infty$. Therefore, the additional restrictions brought by $\bar{A}$ are not relevant.

Now, let ψ be the current parameter of the MCMC sequence. Then, we draw a candidate $\tilde{\psi}$ for the next iteration under the linear inequality constraints:

$$\tilde{\psi} \sim N(\psi, \Omega)1(\bar{A}\psi \geq \bar{a})$$

Let $u := \tilde{\psi} - \psi$. Then,

$$\bar{A}u = \bar{A}\tilde{\psi} - \bar{A}\psi \sim N(0, \bar{A}\Omega\bar{A}')1(\bar{A}u \geq \bar{a}^* := \bar{a} - \bar{A}\psi)$$

Let C be the cholesky decomposition of $\bar{A}\Omega\bar{A}'$. Then, it can be shown that $\bar{A}u = C\varepsilon$ with $\varepsilon \sim N(0, I)1(\varepsilon \geq \underline{\varepsilon})$. The lower bounds are given by

$$\underline{\varepsilon}_j = c_{j,j}^{-1}\left(a_j^* - \sum_{i=1}^{j-1} c_{j,i}\varepsilon_i\right)$$

for $j > 1$ and $\underline{\varepsilon}_1 = c_{1,1}^{-1}\bar{a}_1^*$ where $c_{i,j}$ is the (i, j) element of C. We draw ε's recursively from the truncated standard normal distribution. Then, $\tilde{\psi} = \psi + \bar{A}^{-1}C\varepsilon$.

D More Discussion on Monte Carlo Studies

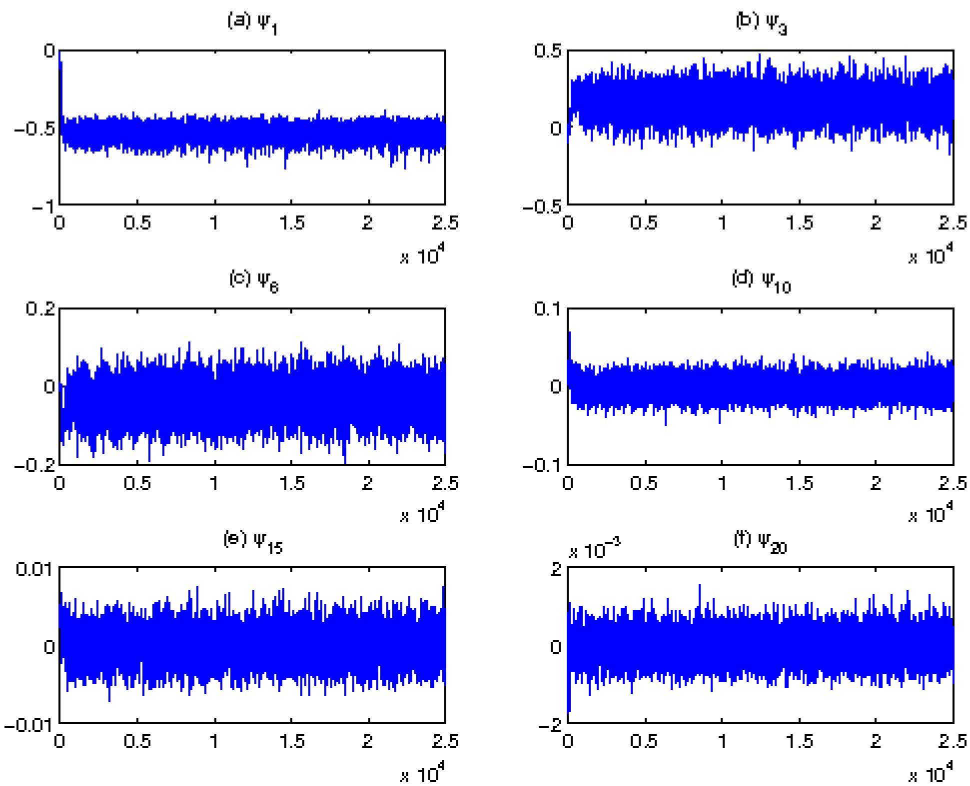
D.1 Implementation of BSL

For $\tilde{f}$ we use the uniform $[0, 1]$ and employ the Legendre polynomials to construct $\{\phi\}$. Especially, we use 20 components. Then, the AM algorithm iterates 500,000 times and we choose every 20 parameters from the last 200,000 iterations to compute the predictive density $\hat{f}_{BSL}(v|y) := E_\theta[f(v|\theta)|y]$ and choose the reserve price maximizing the future revenue. Figure 9 to 12 plots some AM outputs for each Monte Carlos studies. (We record every 20 iterations.) For each iteration, $R = 10,000$ bids are simulated. The bid space is discretized into twenty bins with equal size.

For each data generating process, we use 1,000 Monte Carlo replications. If we run the BSL for each replication separately, it would be very time consuming. In this reason, we implement our method only for the first replication. Then, we estimate the predictive valuation density using an importance sampling method.

Specifically, the predictive valuation density estimate for j-th replication is estimated by $\hat{f}_{BSL}(v|y^j) := \frac{\sum_{s=1}^S w_j(\theta_s)f(v|\theta_s)}{\sum_{s=1}^S w_j(\theta_s)}$ where S is the number of random parameters drawn from the posterior of the first Monte Carlo replication and $w_j(\theta)$ is the ratio of target density (the j-th posterior) and the source density (the 1st posterior). That is, $w_j(\theta) = \frac{p(u,\theta|y^j)}{p(u,\theta|y^1)} = \frac{p(\theta)\hat{L}(\theta|y^j)}{p(\theta)\hat{L}(\theta|y^1)} = \prod_{d=1}^D \{\hat{p}_d(\theta)\}^{y_d^j - y_d^1}$ and y^j is the histogram implied by the discretization and the j-th sample for $j = 2, \cdots, 1,000$. Similarly, we conduct the Bayesian decision method to compute the reserve price for last 999 replications.

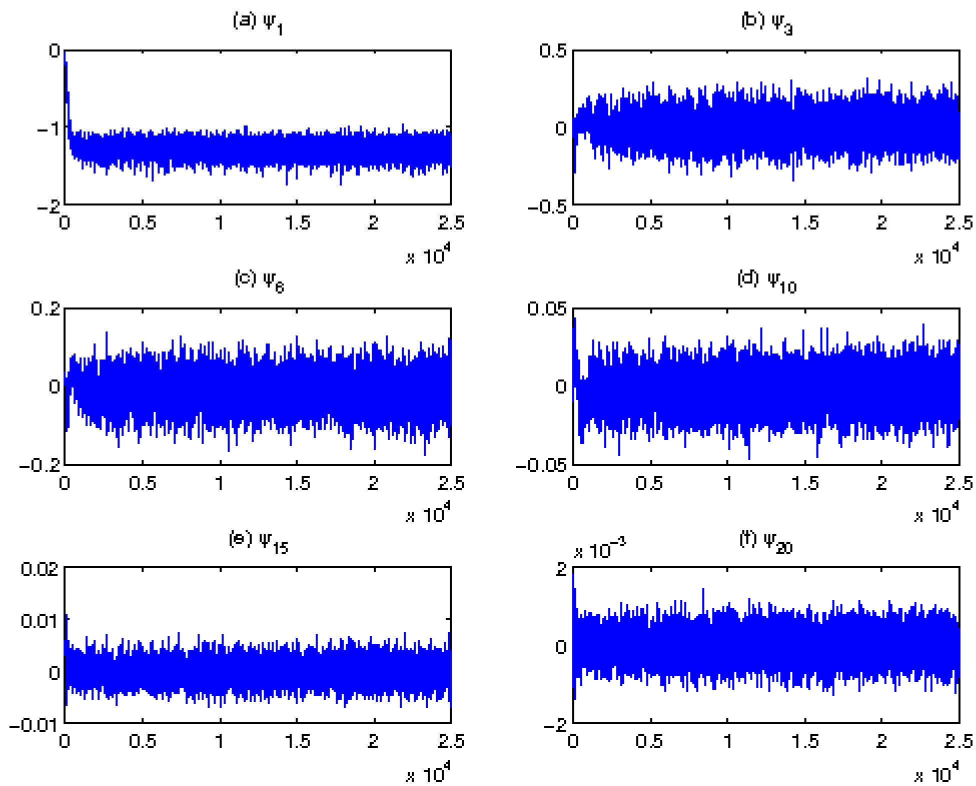
Figure 9: The AM outputs for Log-Normal



D.2 Specification for GPV

We employ the estimation procedure that Guerre, Perrigne, and Vuong (2000) use for their Monte Carlo study. They take the triweight kernel $K(u) := \frac{35}{32}(1-u^2)^3 \cdot 1\{|u|<1\}$. Then, the bid density is given by $\widehat{g}(b|h_g) := \frac{1}{h_g TN} K\left(\frac{b-b_{i,t}}{h_g}\right)$ using the bandwidth $h_g = 1.06 \cdot \text{stdv}(z) \cdot (TN)^{-1/5}$. The pseudo values are computed by $\hat{v}_{i,t} = \widehat{\beta^{-1}}(b_{i,t}) = b_{i,t} + \frac{\widehat{G}(b_{i,t})}{(N-1)\widehat{g}(b_{i,t})}$ only for $b_{i,t} \in [\min(z)+h_g, \max z - h_g]$. Finally, they estimate the valuation density using $\widehat{f}(v|h_g,h_f) := \frac{1}{h_f|\{\hat{v}\}|} K\left(\frac{v-\hat{v}_{i,t}}{h_f}\right)$ with $h_f = 1.06 \cdot \text{stdv}\left(\{\hat{v}\}\right) \cdot \left(|\{\hat{v}\}|\right)^{-1/5}$.

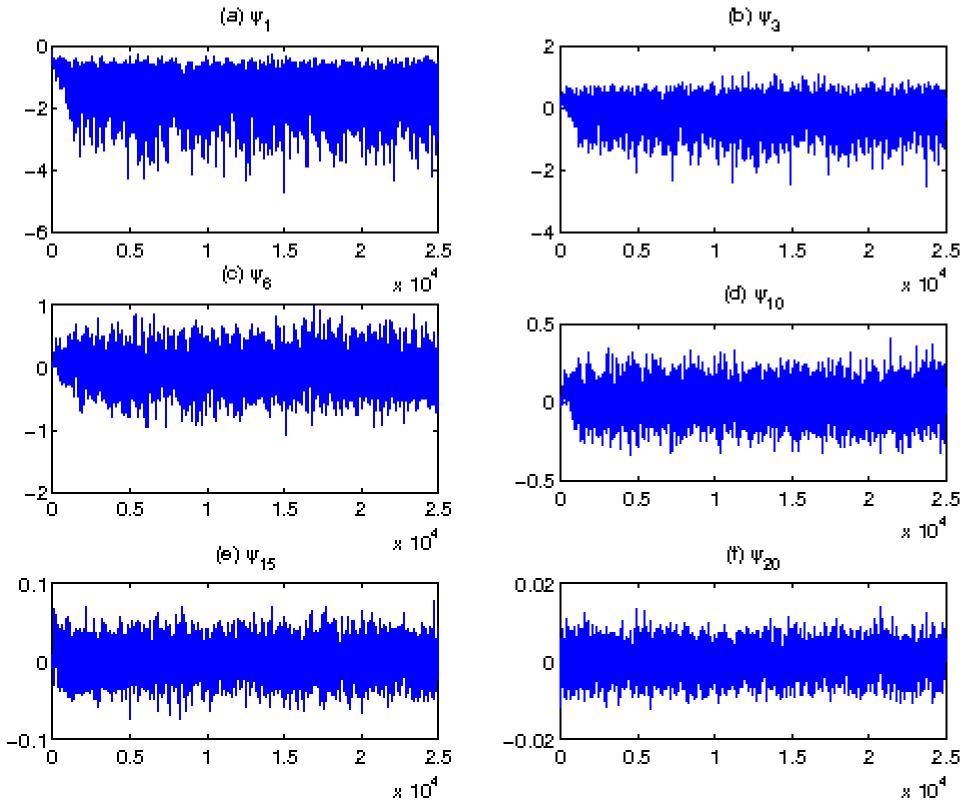
Figure 10: The AM outputs for Exponential



D.3 Specification for Oracle GPV

The bandwidths for Oracle GPV are given by $(h_g^*, h_f^*) := \arg\min_{(h_g, h_f)} \int_0^1 \left\{\widehat{f}(x|h_g, h_f) - f(x)\right\}^2 dx$. We run a grid search to solve this optimization problem. (h_g^*, h_f^*) does not imply a monotone inverse bidding function. Note that the smallest bandwidth for strictly monotone $\widehat{\beta^{-1}}$ is typically too large. Therefore, it leads to too flat bid density estimate which does not fit the data well. Instead, Oracle GPV chooses a very small h_g^* and a large h_f^*: this h_g^* implies a very noisy $\widehat{\beta^{-1}}$ which covers the true inverse bidding function so that the pseudo values from this $\widehat{\beta^{-1}}$ look roughly distributed as the true DGP. Then, the large h_f^*

Figure 11: The AM outputs for Nonsmooth



provides a smooth density estimate over these pseudo values.

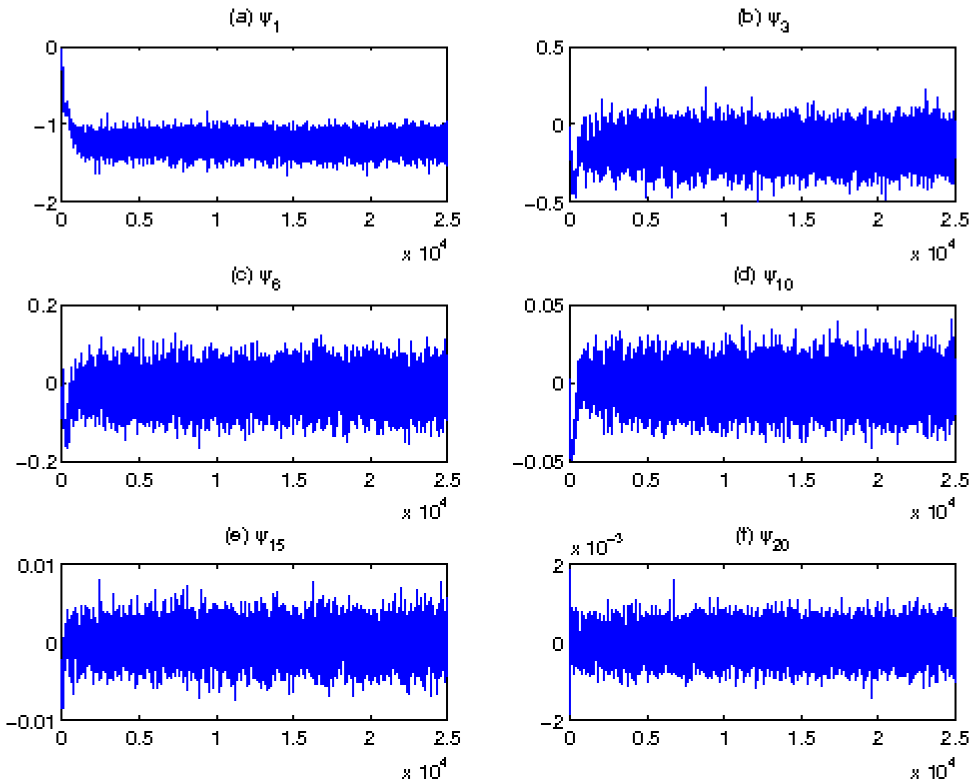
E Appendix to Wildcat Auction Analysis

E.1 Specification for $f(\cdot|\theta)$

we employ $(l,k) = (4,10)$. Then, ψ has 91 components and (9) is two time differentiable.[26] This is differentiable enough for the affiliation condition to be written as

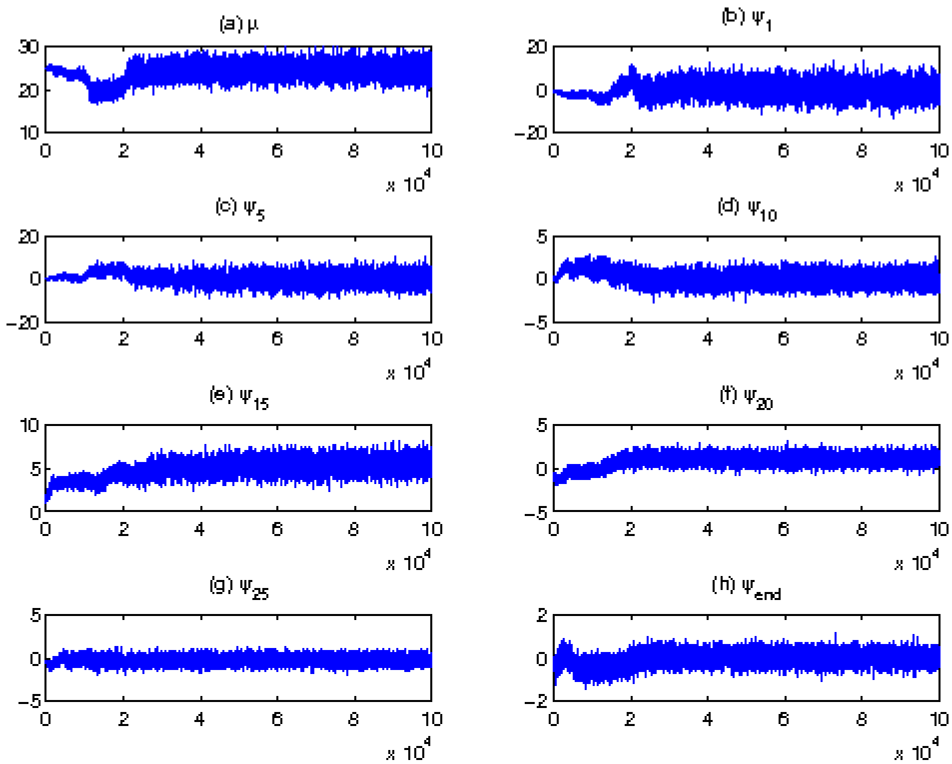
[26] $|\psi| = \frac{|I|(|I|+1)}{2} = 91$ for which $|I| = 2m+k+1 = 2(1)+10+1 = 13$. The l-th derivative does not exist at each grid point $\frac{i}{k}$ for $i \in I$.

Figure 12: The AM outputs for Wildcat-like



(10). We judge this specification to be sufficiently flexible and computationally practical. We iterate the AM algorithm 100,000,000 times and throw away first 60,000,000 of them. Then, we take every 4,000 iterations to make the posterior inference. Hence, $S = 10,000$. Figure 12 plots the AM algorithm for some parameters. (We record every 1,000 iteration.)

Figure 13: The AM outputs for Wildcat-like



E.2 Computation

Now, we discuss the computation of (13). We consistently estimate (14) using

$$\widehat{\Pi}(\theta,\rho) := \frac{1}{\tilde{R}}\sum_{r=1}^{\tilde{R}} \left\{\beta(\tilde{v}_{(2),r}|\rho,\theta)\cdot 1\left(\tilde{v}_{(1),r} > \rho\right) + \rho\cdot 1\left(\tilde{v}_{(1),r} < \rho < \tilde{v}_{(2),r}\right)\right\} \tag{19}$$

where $\left\{(\tilde{v}_{1,r},\tilde{v}_{2,r})\right\}_{r=1}^{\tilde{R}} \sim f(\cdot,\cdot|\theta)$. Hence, we approximate the Bayes action (13) with

$$\widehat{\rho}_B(y) := \arg\max_{\rho\in\mathcal{A}} \frac{1}{S}\sum_{s=1}^{S}\widehat{\Pi}(\theta_s,\rho) \tag{20}$$

where $\{\theta_s\}_{s=1}^S$ are the MCMC output.

Note that for the AM algorithm we have used new simulation draws at each iteration to obtain the unbiased likelihood estimates. However, we use a fixed uniform draw $\{(\tilde{u}_{1,r}, \tilde{u}_{2,r}\}_{r=1}^{\tilde{R}}$ to compute (19) for each ρ and θ, because, otherwise, the sample mean in (20) would not be smooth. Now, to evaluate (20), we employ the algorithm as follows. For a fixed θ

1. Compute $\{(\tilde{v}_{1,r}, \tilde{v}_{2,r})\}_{r=1}^{\tilde{R}}$ using an inverse CDF method:
   $\tilde{v}_{1,r} = F^{-1}(\tilde{u}_{1,r}|\theta)$ and $\tilde{v}_{2,r} = F^{-1}(\tilde{u}_{2,r}|\tilde{v}_{1,r}, \theta)$ with the marginal CDF, $F(v_1|\theta) := \int_0^{v_1}\int_0^{\infty} f(s,t|\theta)dtds$, and the conditional CDF, $F(v_2|v_1,\theta) := \int_0^{v_2} \frac{f(v_1,s|\theta)}{\int_0^{\infty} f(v_1,t|\theta)dt}ds$.

2. Compute $\{\beta(\tilde{v}_{(2),r}|\theta)\}_{r=1}^{\tilde{R}}$ using the recursive method we have employed for the AM algorithm.
   Note that $\tilde{v}_{(2),r} > \tilde{v}_{(1),r}$ for each $r = 1,\ldots,\tilde{R}$ and $\beta(v|\theta) := \beta(v|0,\theta)$

3. For each $\rho \in \mathcal{A}$, compute $\{\beta(\tilde{v}_{(2),r}|\rho,\theta)\}_{r=1}^{\tilde{R}}$ using $\beta(v|\rho,\theta) = \{\beta(v|\theta) + L(\rho|v,\theta)(\rho - \beta(v|\theta))\}\cdot 1(v > \rho)$ following Li, Perrigne, and Vuong (2003) and construct (19).

We do this for each θ_s, $s = 1,\ldots,S$. Then, solving (20) is easy.

Only the larger valuations $\{\tilde{v}_{(2),r}\}_{r=1}^{\tilde{R}}$ could be used for step 2 above. However, we compute all the bids $\left\{\left(\tilde{b}_{1,r}(\theta), \tilde{b}_{2,r}(\theta)\right)\right\}_{r=1}^{\tilde{R}}$ with $\tilde{b}_{i,r}(\theta) = \beta(\tilde{v}_{i,r}|\theta)$ for $i = 1,2$ and $r = 1,\ldots,\tilde{R}$. This is useful to estimate the predictive bid density. That is, we may estimate the marginal of (3) using

$$\widehat{g}(b|f(\cdot|\theta)) := \frac{1}{2\tilde{R}h_m}\sum_{r=1}^{\tilde{R}}\sum_{i=1}^{2} K\left(\frac{b - \tilde{b}_{i,r}(\theta)}{h_m}\right) \tag{21}$$

Then, the marginal predictive bid density is estimated by

$$\widehat{g}(b|y) := \frac{1}{S}\sum_{s=1}^{S}\widehat{g}(b|f(\cdot|\theta_s)) \tag{22}$$

In addition, to estimate the joint predictive bid density, we first estimate

$$\tilde{g}(b_1, b_2|y) := \frac{1}{S}\sum_{s=1}^{S}\left\{\frac{1}{\tilde{R}h_j}\sum_{r=1}^{\tilde{R}} K\left(\frac{b_1 - \tilde{b}_{1,r}(\theta_s)}{h_j}\right) K\left(\frac{b_2 - \tilde{b}_{2,r}(\theta_s)}{h_j}\right)\right\}$$

with a kernel K and some bandwidths h_m and h_j and symmetrize this using

$$\widehat{g}(b_1, b_2|y) = \frac{1}{2}\left\{\tilde{g}(b_1, b_2|y) + \tilde{g}(b_2, b_1|y)\right\} \tag{23}$$

We employ $\tilde{R} = 1,500$.

References

[1] Andrieu, C., A. Doucet, and R. Holenstein (2007), "Particle Markov chain Monte Carlo," *Unpublished paper: Department of Mathematics, University of Bristol.*

[2] Andrieu, C., A. Doucet, and G. Roberts (2007), "The expected auxiliary variable method for Monte Carlo simulation," *Unpublished paper*

[3] Beresteanu, A. (2007) "Nonparametric Estimation of Regression Functions under Restrictions on Partial Derivatives," *Working Paper*

[4] Campo, S., Perrigne, I., and Vuong, Q. (2003), "Asymmetry in First-Price Auctions with Affiliated Private Values," *Journal of Applied Econometrics,* **18**: 179–207

[5] Donald, S. and Paarsch, H. (1993), "Piecewise Pseudo-Maximum Likelihood Estimation in Empirical Models of Auctions," *International Economic Review,* **34** : 121–148.

[6] Donald, S. and Paarsch, H. (1996), "Identification, Estimation, and Testing in Parametric Empirical Models of Auctions within the Independent Private Values Paradigm," *Econometric Theory,* **12** : 517–567.

[7] Flury, T., and Shephard, N. (2008) "Bayesian Inference Based only on Simulated Likelihood: Particle Filter Analysis of Dynamic Economic Models," *Working Paper.*

[8] Fernandez-Villaverde, J. and J. F. Rudio-Ramirez (2007) "Estimating macroeconomics models: a likelihood approach," *Review of Economic Studies,* **74** : 1059–1087.

[9] Fernandez-Villaverde, J., J. F. Rudio-Ramirez, and M. S. Santos (2006) "Convergence properties of the likelihood of computed dynamic models," *Econometrica,* **74**: 93–119.

[10] Fritsch, F. N. and R. E. Carlson, (1980) “Monotone Piecewise Cubic Interpolation,” *SIAM Journal of Numerical Analysis,* **17**: 238–246

[11] Gallant, A.R., and Nychka, D.W. (1987) “Seminonparametric maximum likelihood estimation,” *Econometrica,* **55,** 363–390.

[12] Geweke, J. (1995), “Bayesian Inference for Linear Models Subject to Linear Inequality Constraints” *Working Paper*.

[13] Guerre, E., Perrigne, I., and Vuong, Q. (2000), “Optimal Nonparametric Estimation of First-Price Auctions,” *Econometrica,* **68**: 525–574.

[14] Haario, H., Saksman, E., and Tamminen, J. (2001) “An Adaptive Metropolis Algorithm,” *Bernoulli,* **7** : 223–242.

[15] Hajivassiliou, V., and McFadden, D. (1998), “The Method of Simulated Scores for the Estimation of LDV Models,” *Econometrica,* **66**: 863 – 896.

[16] Hendricks, K., Pinkse, J., and Porter, R. (2003) “Empirical Implications of Equilibrium Bidding in First-Price, Symmetric, Common Value Auctions,” *Review of Economic Studies,* **70**: 115–145.

[17] Hirano, K. and Porter, J. (2003) “Asymptotic Efficiency in Parametric Structural Models with Parameter-Dependent Support,” *Econometrica,* **71**: 1307–1338.

[18] Keane, M. (1990), “Four Essays in Empirical Macro and Labor Economics” Ph.D. dissertation, Brown University.

[19] Kim, D. (2008) “Auction Design using Bayesian Methods,” *Working Paper: University of Arizona,* **08-11**.

[20] Laffont,J., Ossard, H., and Vuong, Q. (1995) “Econometrics of First-Price Auctions,” *Econometrica,* **63** : 953–980

[21] Laffont,J. and Vuong, Q., (1996) “Structural Analysis of Auction Data,” *The American Economic Review,* **86**, : 414–420.

[22] Li,T., Perrigne, I., and Vuong, Q., (2002) "Structural Estimation of the Affiliated Private Value Auction Model," *RAND Journal of Ecoomics,* **33**, : 171–193.

[23] Li,T., Perrigne, I., and Vuong, Q., (2003) "Semiparametric Estimation of the Optimal Reserve Price in First-Price Auctions," *Journal of Business & Economic Statistics,* **21**(1), : 53–64.

[24] Li,T. and Vuong, Q., (1997) "Using All Bids in Parametric Estimation of First Price Auctions," *Economics Letters,* **55** : 321–325.

[25] Milgrom, P. R., and Weber, R. J. (1982), "A Theory of Auctions and Competitive Bidding," *Econometrica,* **50**, 1089–1122.

[26] Newey, N. K., and McFadden, D. (1994) "Large Sample Estimation and Hypothesis Testing," *Handbook of Econometrics,* **4**, 2111 – 2245

[27] Paarsch, H. J. (1997), "Deriving an Estimate of the Optimal Reserve Price: An Application to British Columbian Timber Sales," *Journal of Econometrics,* **78**, 333–357.

[28] Riley, J., and Samuelson, W. (1981), "Optimal Auctions," *American Economic Review,* **71**, 381–392.

[29] Verdinelli, I. and Wasserman, L. (1998), "Bayesian Goodness-Of-Fit Testing Using Infinite-Dimensional Exponential Families," *The Annals of Statistics,* **26**, 1215–1241.

[30] Wilson, R., (1977), "A Bidding Model of Perfect Competition," *Review of Economic Studies,* **44**: 511–518.